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MAXIMUM POSSIBLE NUMBER OF PARETO OPTIMUM PATHS

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Abstract

Author try to explain real life applications, dealing with the networks, such as transportation networks, communication networks and pipeline distribution require the computation of best or shortest paths from one node to another, called Shortest Path Problem (SPP). The primary aim of the network models is to optimize the performance with respect to pre defined objectives.

Keywords: Network, Pareto Optimum Path.

Introduction

Many real life applications, dealing with the networks, such as transportation networks, communication networks and pipeline distribution require the computation of best or shortest paths from one node to another, called Shortest Path Problem (SPP). The primary aim of the network models is to optimize the performance with respect to pre defined objectives.

When only one objective (criteria) is considered in the network, SPP is called a Single Objective Shortest Path Problem (SOSPP). SOSPP appears in applications as a sub problem for which several efficient algorithms are available in literature. A Multiple Objective Shortest Path Problem (MOSPP) in a network consists of more than one objective. Multiple objectives such as optimization of cost, time, distance, delay, risk, reliability, quality of service and environment impact etc. may arise in such problems.

Maximum number of Pareto optimal paths of a MOSPP in a network, is very much useful in finding the maximum number of iterations and in determining the complexity of a particular algorithm.

It is proved here that the maximum number of Pareto optimal paths of any MOSPP in a completely connected network with nodes, in the worst case, is 1+(n-2)+(n-2)(n-3)+...+(n-2)!+(n-23)! and it lies between 2[(n-2)!] and 3[(n-2)!].

Also, a recurrence relation is developed to find the maximum number of Pareto optimal paths of any MOSPP in the worst case as p(n) = (n-2) p(n-1)+ 1, p(2) = 1.

Terminology

Network: A network represented by G = (V, E, d) consists of the set V of n nodes (vertices), the set E ⊆ V × V of edges (arcs) and a function d defined as d: E→ Rm, that is, each edge (I, j) € E is associated with a vector weight dij = (dij1, dij2, …, dijm). Here, m represents the number of objectives.

If m = 1, the network is called single objective network and if m > 1, it called multi objective network. If each edge has direction, then it is called directed network.

Completely Connected Network

Completely connected network is a network, in which every ordered pair of nodes is having a directed edge.

Path and Loop-Less Path: A path between the nodes i and j is a sequence of continuous edges that connects nodes i and j.

We can also represent a path by the sequence (vi, … vj) of its nodes. Lf the sequence of nodes of the path is finite, then the path is said to be finite. A path of a network is said to be loop-less, if all its nodes are distinct.

Value of a Path: Value of a path is the sum of the weights of the constituting the path. Let S be the set of all paths between any two nodes.

The value of a path p € S of a MOSPP is a vector function w: S → Rm such that w (p) = (w1 (p), w2(p), …, wm(p)) and each component wr, r € {1, 2, ..., m} is defined as a function wr: S → R. If p is a

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path (v1, v2,…, v11), then $w^r (p) = \sum_{i=1}^{n-1} d^r$ i. i + 1, where r € {1, 2, …, m}.

That is, value of a MOSPP is a vector, each of its components is the sum of the corresponding component of the vector weights of the edges constituting the path.

Order Relation on Rm: Let a = (a1, a2, ..., am) and b = (b1, b2, …, bm) be any two vector elements of Rm then a <Rm b if and only if a1 ≤ b1 for all i = 1, 2, …, m and a1 < b1 for at least one i.

Dominance (<D) of Paths: Let p and q be two paths S. p <D q (p dominates q or q is dominated by p) if and only if w(p) <Rm w (q).

Non-Dominated or Pareto Minimum Path: For an MOSPP with S the set of all paths between two specific nodes of a network, a path p* € S is said to be Pareto minimum if there does not exist any other path p € S such that p < p*.

Maximum Number of Pareto Optimal Paths

Theorem

The maximum number of Pareto minimum paths from node 1 to any other node t, of any MOSPP, in a completely connected network without multiple edges and self-loops is between 2[(n-2)!] and 3[(n-2)!], where n is the number of nodes.

Proof:

Consider a completely connected directed network G = (V, E), where V = {1, 2, …, n} is the set of n nodes and E ⊆ V × V is the set of edges. Without loss of generality, we assume that every loop-less path from node 1 to any other node t is Pareto minimum. Therefore, to generate all Pareto minimum paths it is enough to generate all distinct loops-less path from node 1 to node t.

As our aim is to find all loops-less paths from initial node 1 to all other node t, all edges whose head node is 1 are removed from the network.

Paths from node 1 to node t with only one edge, that is, paths not passing through any intermediate node is (1, t).

Hence, number of paths with only one edge is 1. Paths with two edges, that is, paths passing through only one intermediate node are (1, 2, t), (1, 3, t), …, (1, n, t). Hence, number of paths with two edges, is nothing but permutation of (n-2) nodes (excluding 1 and t) taken one at a time, is Pn-2 1 = (n-2). Similarly, number of paths with three edges, (that is, number of paths passing through two intermediate nodes).

Which is nothing but permutation of (n-2) nodes taken 2 nodes at a time, is Pn-2, 3 = (n-2) (n-3). Similarly, number of paths with (n-2) edges, is the permutation of (n-2) nodes taken (n-3) nodes at a time, is Pn-2, n-3 = (n-2)(n-3) … (3)(2) = (n-2)!. Lastly, the number of paths with (n-1) edges is the permutation of (n-2) nodes taken (n-2) nodes at a time, is Pn-2, n-3 = (n-2) (n-3) … (2)(1) = (n-2)!. Therefore, total maximum number of distinct loop-less paths from node 1 to node t is

= Pn-2.0 + Pn-2, 1 + Pn-2, 2 + … + Pn-2, n-2
= 1 + (n-2) + (n-2) (n-3) + … + (n-2)! + (n-2)!
= (n-2)! {1/(n-2)! + 1/(n-3)! + … + 1/3! + ½! + 1+1}
= (n-2)! {1/(n-2)! + 1/(n-3)! + … + 1/3! + ½! + 2},

where n≥2.

The sum of the terms inside the bracket lies between 2 and 3. Hence, maximum number of Pareto minimum paths from node 1 to any other node t lies between 2[(n-2)!] and 3[(n-2)!].

Theorem

The above result in theorem 1 obeys the recurrence relation, p(n) = (n-2) p(n-1) +1 and p(2) = 1, where p(n) is the maximum number of distinct loop-less paths from initial node 1 to any other node t and n is the number of nodes of the completely connected network.

Proof:

For n = 2 the theorem is true, as there is only one edge from node 1 to node 2. Hence, there will be only one path, that is, p (2) = 1.

Let us assume that the theorem is true for an n node network, that is, p (n) = (n-2) p (n-1) +1. Hence: the maximum number of distinct loop-less paths from node 1 to each of the (n-1) nodes is P (n). To prove: p (n+1) p(n) + 1.

Let us suppose that one node is included with the existing network in such a way that the resulting network is a completely connected network with (n+1) nodes.

Hence, each of the (n-1) set of p(n) paths will also be a path from node 1 to the new node and there will be one more path due to the edge (1, new node). Therefore, total number of distinct loop-less paths from node 1 to new node in a (n+1) node network will be (n-1) p(n) + 1, which is nothing but p(n+1).

Note: Using this result recursively, we can determine the maximum number of distinct loop-less paths from a specific starting node to any other node of a completely connected network.

Examples:

1. Completely connected network with 2 nodes {1, 2} has only one path from node 1 to node 2.
2. Completely connected network with 3 nodes {1, 2, 3} has the following 2 distinct loop-less paths from node 1 to node 3 at the most: {<1, 3>, <1, 2, 3>}
3. Completely connected network with 4 nodes {1, 2, 3, 4} has the following 5 distinct loop-less paths from node 1 to node 4 at the most {<1, 4>, <1, 2, 4>, <1, 3, 4>, <1, 2, 3, 4>, <1, 3, 2, 4>}
4. Completely connected network with 5 nodes {1, 2, 3, 4, 5} has 16 distinct loop-less paths from node 1 to node 5 at the most.

In all the above cases, the maximum number of distinct loop-less paths lies between 2[(n-2)!] and 3[(n-2)!] and they obey the recurrence relation, p(n) =(n-2) p(n-1)+1, p(2)= 1.

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